

**ArsDigitaUniversity**  
**Month8:TheoryofComputation**  
**ProfessorShaiSimonson**

**ProblemSet5**

**1. ShortAnswers:Name thatLanguage**

Foreachofthelanguagesbelow,indicatethesmallestcomplexityclassthatcontainsit.(i.e.Regular, DeterministicContextFree,ContextFree,TuringMachines(Recursive)).Assumeanalphabetof $\{0,1\}$  unlessotherwise specified.Youdonotneedtoproveyouranswers.

- a.  $0^n 1^m 0^p 1^q$ , where  $n+m=p+q$ , and  $n, m, p, q > 0$ .
- b.  $0^n 1^m 0^n$ , where  $n, m > 0$ .
- c.  $0^n 1^m 0^p 1^q$ , where  $n, m, p, q > 0$ .
- d. Thesetofstringsoveralphabet $\{0,1,2\}$ withanequalnumberof0'sand2's,oranequalnumberof 0'sand1's.
- e.  $0^m$  over the alphabet  $\{0\}$  where  $m$  is of the form  $2k+1$ ,  $k > 0$ .
- f. Thesetofstringswith $3n$ 0'sand $4m$ 1's, form  $n > 0$ .
- g. Thesetofstringsthathaveat leasttentimesasmany0'sas1's.
- h. Thesetofstringsthatareeitheroddlengthorcontain5consecutive1's.
- i.  $0^m 10^{m!}$ ,  $m > 0$ .
- j. Thesetofstringsoveralphabet $\{0,1,2\}$ wherethenumberof1'sequalsthenumberof2's,and every0isfollowedimmediatelybyatleastone1.

**2. Optional:MoreTuringMachineDesign**

- a. DesignaTMprogramtotakeabinaryintegernasinput,andreturnthebinarystringwithvalue  $n+1$ .Youmayerasetheinputifyouwant.
- b. DesignaTMsubroutinewhichtakesabinarystringandcopiesittotherightoftheinputwitha\$ infront.Thatis,itturns $x$ into $\$x$ .
- c. DesignaTMsubroutineturns $\$x$ into $\$x\$x$ .
- d. DesignaTMtoacceptstringsoftheform $ww$ .

**3. AnUndecidabilityProblem.**

ProvethattheproblemofdeterminingifthelanguagegeneratedbytwoCFG'sareequalis undecidable.Youmayrefertoanyresultdiscussedinclass.

**4. IsitR.E.orisitnot?**

Foreachofthefollowinglanguages,statewhetherthelanguageisorisnotrecursivelyenumerableand whetherthecomplementofthelanguageisorisnotrecursivelyenumerable.Givesomejustification foryouranswers.

- a. ThelanguageofallTM'sthatacceptnothing.
- b. ThelanguageofallTM'sthataccepteverything.
- c. ThelanguageofallTM'sthatacceptRegularsets.
- d. ThelanguageofallPDA'sthataccepteverything.
- e. ThelanguageofallCFG'sthatareambiguous

**5. A Refutation of the Halting Problem?**

Consider the language of all  $TM'$ 's that given no input eventually write an  $\epsilon$ -blank symbol on their tapes. Explain why this set is decidable. Why does this not contradict the halting problem?

**6. The Post Correspondence Problem for One  $\Sigma$ -Character Strings.**

Prove that the PCP problem is decidable for strings over the alphabet  $\{0\}$ .

**7. Extra Credit: Even Oracles Sometimes Need Oracles.**

(Text: 6.22) Let

$$Z = \{ \langle M, w \rangle \mid M \text{ is an oracle TM and } M^{A_{TM}} \text{ accepts } w \}.$$

Use a proof by diagonalization to show that an oracle  $TM$  with an oracle for  $A_{TM}$  can't decide  $Z$ .

**8. Extra Credit: Variations on the theme of 3SAT.**

- a. Prove that the 3SAT variation where each variable  $x_i$  and  $\neg x_i$  appear in exactly two clauses is still NP-Complete.
- b. Prove that the 3SAT variation where each variable  $x_i$  and  $\neg x_i$  appear in exactly one clause is solvable in polynomial time. (Hint: Think of the polynomial algorithm for 2SAT).

**9. Satisfiability for DNF Formulas is in P.**

Prove that the problem of determining whether there is a T/F assignment that makes a given *disjunctive normal formula* true can be solved in polynomial time. How do you explain this in light of the fact that any formula in conjunctive normal form can be converted to one in disjunctive normal form, and the satisfiability of CNF formulas is NP-Complete?

**10. A Punchcard Puzzle that is NP Complete.**

(Text: 7.26). You are given a box and a collection of  $n$  cards. Because of the pegs in the box and notches on the cards, each card will fit in the box of either two ways. Each card contains two columns of holes, some of which may not be punched out. The cards can be flipped about the vertical axis so that the columns are interchanged. The puzzle is solved by placing all the cards in the box so as to completely cover the bottom of the box, (i.e., every hole position is blocked by at least one card that has no hole there.) Let

$$PUZZLE = \{ \langle c_1, \dots, c_k \rangle \mid \text{each } c_i \text{ represents a card and this collection of cards has a solution.} \}$$

Show PUZZLE is NP-complete. (See the text for an illustration of the cards.)

**11. PSPACE Hard implies NP Hard**

(Text: 8.6). Show that any PSPACE-hard language is also NP-hard.

## 12. AT IC-TAC-TOE-Like GamethatisinPSPACE

(Text:8.10).TheJapanesegamego-mokuisplayedbytwoplayers,“X”and“O”,ona19x19grid. Playerstaketurnsplacingmarkers,andthefirstplayertoachieve5ofhismarkersconsecutivelyina row,column,ordagonal,isthewinner.Considerthisgamegeneralizedtoanbynboard.Let

$$GM = \{ \langle P \rangle \mid P \text{ is a positioning generalized go-moku,} \\ \text{where player "X" has a winning strategy} \}$$

A position means a board with markers on it, such as may occur in the middle of a play of the game. Show that  $GM$  is in PSPACE.

## 13. APunchcardPuzzlethat is PSPACE Complete.

(Text:8.14).Considerthefollowingtwo-personvariationofthelanguagePUZZLEthat is described in problem d. above. Each player starts with an ordered stack of puzzle cards. They take turns placing them in order in the box and may choose which side faces up. Player I wins if, in the final stack, all hole positions are blocked, and Player II wins if some hole position remains unblocked. Show that the problem of determining which player has a winning strategy for a given starting configuration of the cards is PSPACE-complete.

## 14. Extra Credit: Regular Expression Equivalence is in PSPACE

(Text:8.16).Let  $EQ_{\text{REX}} = \{ \langle R, S \rangle \mid R \text{ and } S \text{ are equivalent regular expressions} \}$ . Show that  $EQ_{\text{REX}}$  is in PSPACE.

## 15. Extra Credit: Kleene Star Preserves P

(Text:7.13).Showthat  $P$  is closed under the Kleene star operation. (Hint: on input  $y = y_1 \dots y_n$  for  $i$  in  $\Sigma$ , build a table indicating for each  $i \leq j$  whether the substring  $y_i \dots y_j$  is in  $A^*$  for any  $A$  in  $P$ .)

## 16. Extra Credit: The Game of Nim and Logarithmic Space

(Text:8.21).Thegameofnimisplayedwithacollectionofpilesofsticks.Inonemoveaplayermay remove any non-zero number of sticks from a single pile. The players alternately take turns making moves. The player whose move leaves the very last stick loses. Say that we have a game position in NIM with  $k$  piles containing  $s_1, \dots, s_k$  sticks. Call the position balanced if, when each of the numbers  $s_i$  is written in binary and the binary numbers are written as rows of a matrix aligned at the lower order bits, each column of bits contains an even number of 1's. Prove the following two facts:

- Starting in an unbalanced position, a single move exists that changes the position to a balanced one.
- Starting in a balanced position, every single move changes the position to an unbalanced one.

Let

$$NIM = \{ \langle s_1, \dots, s_k \rangle \mid \text{each } s_i \text{ is a binary number and Player I has a winning strategy in the NIM game starting at this position.} \}$$

Using the preceding facts about balanced positions to show that NIM is in L, the class of languages that are decidable in logarithmic space on a deterministic Turing Machine.

**17. Extra Credit: Two Counters Are All You Really Need In Life**

Prove that two counters are enough to simulate a Turing Machine. (Hint: prove that two counters can simulate a stack, then prove that two counters can simulate four counters).

**18. Extra Credit: Infinite Recursive Sets Hiding In R.E. Sets**

Show that every infinite recursively enumerable set has an infinite recursive subset. (Hint: Prove that a TM exists that *generates* the r.e. set. That is, it starts with an empty tape and keeps printing strings in the set. Consider the set of all strings  $x$  which are among the first  $2^{|x|}$  strings generated by this TM. Prove that this set is infinite and recursive.)