

PROBLEM SET 16 SOLUTIONS

by Michael Allen

- (1) Check that $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ form a basis for \mathbb{R}^3 . Transform this basis into an orthonormal basis using the Gram-Schmidt algorithm. Check that the resulting vectors are indeed orthogonal!

Answer: The three vectors will form a basis for \mathbb{R}^3 iff they are linearly independent. To determine this, we can simply place them into the columns of a matrix, and reduce it to find the pivots.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since there are three pivots, the vectors are linearly independent and form a basis for \mathbb{R}^3 .

Now, we can use Gram-Schmidt to find an orthogonal basis:

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{6} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

And then normalize:

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

- (2) Check that the vectors $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$ form a basis of \mathbb{R}^4 . Use the Gram-Schmidt algorithm to make this into an orthonormal basis.

Answer: As in the last problem, we can determine if the vectors form a basis by putting the vectors into the columns of a matrix and counting the pivots.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are four pivots, so everything is okay. Now, we use Gram-Schmidt on them (or just do it by inspection), and we find that the orthonormal basis is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- (3) Consider the orthonormal vectors

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

in \mathbb{R}^3 . Find some other vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

such that v_1 , v_2 , and b are a basis of \mathbb{R}^3 . Then use the Gram-Schmidt algorithm to make your basis into an orthogonal one.

Answer: As long as we choose a vector which is linearly independent of the two given, we will have a valid basis, so there are many possible answers. So, let us choose $b = (1, 0, 0)$.

Then, when we perform Gram-Schmidt on the basis, we will get the following:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- (4) Check that the matrix

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

is an orthogonal matrix by checking that $Q \cdot Q^T = I$. Also, check that $\|Q \cdot v\| = \|v\|$ for the vector $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Answer:

$$\begin{aligned}
Q \cdot Q^T &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \cos \theta \sin \theta \\ \cos \theta \sin \theta - \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
\|Q \cdot v\| &= \left\| \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 2 \cos \theta - \sin \theta \\ 2 \sin \theta + \cos \theta \end{bmatrix} \right\| \\
&= \sqrt{5 \sin^2 \theta + 5 \cos^2 \theta} = \sqrt{5} = \sqrt{2^2 + 1^2} = \|v\|
\end{aligned}$$

(5) We have the following theorem.

Theorem 1. *If $\{v_1, \dots, v_n\}$ is an orthonormal basis for \mathbb{R}^n , then for any $v \in \mathbb{R}^n$, we can write*

$$v = c_1 v_1 + \dots + c_n v_n,$$

where $c_i = v \cdot v_i$ ($1 \leq i \leq n$), where \cdot is the dot product.

(a) Use this theorem to write the vector $(3, 2)$ as linear combinations of the vectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \& \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Answer:

$$\begin{aligned}
\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \frac{5}{\sqrt{2}} \\
\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} &= \frac{-1}{\sqrt{2}} \\
\frac{5}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{-1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\end{aligned}$$

(b) Use this theorem to write $(1, 2, -1)$ in terms of the basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Answer: Since the basis is not orthonormal, the theorem does not apply to this problem, and we are stuck.

(6) Consider the two bases of \mathbb{R}^3 ,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The change of basis matrix $M_{\hat{B}}^B$ is

$$M_{\hat{B}}^B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Compute $M_B^{\hat{B}}$.

Answer: $M_B^{\hat{B}}$ is just the inverse of $M_{\hat{B}}^B$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

(7) Now consider the two bases

$$\hat{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and

$$\tilde{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Compute the matrices $M_{\tilde{B}}^{\hat{B}}$ and $M_{\hat{B}}^{\tilde{B}}$.

Answer:

$$M_{\tilde{B}}^{\hat{B}} = \tilde{B}\hat{B}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 \\ -1 & 3 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$M_{\hat{B}}^{\tilde{B}} = M_{\tilde{B}}^{\hat{B}-1} = \frac{1}{4} \begin{bmatrix} 6 & 8 & 6 \\ 5 & 6 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$