PROBLEM SETS 13 SOLUTIONS.

Problem Set 13: Due 22 September 2000.

Reading. *Matrices and Transformations*, none.
Supplementary reading. Strang, sections 3.1–3.2.

(1) **Consider the set $M_2 = \{2 \times 2$ matrices$\}$ as a vector space. Let**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$

   (a) **Name a subspace containing A but not B.**
   All mulitples of the array A form a subspace containing A but not B.
   (b) **Name a subspace containing B but not A.**
   All multiples of the array B form a subspace containing B but not A.
   (c) **Is there a subspace containing A and B but not the 2×2 identity matrix?**
   No. A subspace must be closed under linear combination. If a subspace contains A and B, it contains $\frac{1}{2}A - \frac{1}{3}B$, which is the 2×2 identity matrix.

(2) **Consider $\mathbb{R}^2$ as a vector space. Which of the following are subspaces and which are not? If not, why not?**
   (a) $\{(a, a^2) \mid a \in \mathbb{R}\}$ Not a subspace — consider $(1, 1) + (2, 4) = (3, 5)$; this set is not closed under addition.
   (b) $\{(b, 0) \mid b \in \mathbb{R}\}$ This is a subspace.
   (c) $\{(0, c) \mid c \in \mathbb{R}\}$ This is a subspace.
   (d) $\{(m, n) \mid m, n \in \mathbb{Z}\}$ Not a subspace — not closed under scalar multiplication: consider $\frac{1}{2} \cdot (1, 1)$.
   (e) $\{(d, e) \mid d, e \in \mathbb{R},\ d \cdot e = 0\}$ Not a subspace — not closed under addition: consider $(1, 0) + (0, 1)$.
   (f) $\{(f, f) \mid f \in \mathbb{R}\}$ This is a subspace.

(3) **Show that for some $b \neq 0$, the solution set $\{x \mid Ax = b\}$ does not form a subspace. (Hint: look at Problem set 11, problem number 7.)**
   Suppose that the solution set to $Ax = b$ *always* forms a subset. Then, for any two solutions x_1 and x_2, we have $A(x_1 + x_2) = b$, because subspaces are closed under addition. But then we have $Ax_1 + Ax_2 = b$, which, combined with either $Ax_1 = b$ or $Ax_2 = b$, can be used to conclude that $Ax_1 = 0$ or $Ax_2 = 0$, an absurdity. We conclude that the solution set to $Ax = b$ is not, in general, a subspace.

(4) **Considerthe set $M_n = \{n \times n$ matrices$\}$ as a vector space. Which of the following are subspaces?**
   (a) **The symmetric matrices,** $S = \{A \mid A^{\mathrm{T}} = A\}$ The symmetric matrices are a subspace.

(b) The non-symmetric matrices, $NS = \{A \mid A^{\mathrm{T}} \neq A\}$ The non-symmetric matrices are not a subspace — it is easy to come up with two non-symmetric matrices whose sum is symmetric.

(c) The *skew-symmetric* matrices, $S = \{A \mid A^{\mathrm{T}} = -A\}$ The skew-symmetric matrices are a subspace.

(5) Describe the column spaces of the following matrices.

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} C(C) &= \{c_1(1,2,-1)' + c_2(2,0,3)' : (c1, c2) \in \mathbb{R}^2\} \\ C(D) &= \{c_1(1,2,-1)' + c_2(2,0,3)' : (c1, c2) \in \mathbb{R}^2\}(= C(C)) \end{aligned}$$

Although D has three columns, the column that was added to C to make D is a linear combination of the two coluns of C; the column spaces of the two matrices are identical.

(6) Describe the null-space for the following matrices.

$$E = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \end{bmatrix}$$

$$\begin{aligned} N(E) &= \{c \cdot (2,-4,-3)' : c \in \mathbb{R}\} \\ N(F) &= \{0\} \\ N(G) &= \{c \cdot (10,1,3)' : c \in \mathbb{R}\} \end{aligned}$$

(7) Let P be the plane in $\mathbb{R}^3$ defined by the equation

$$x - y - z = 3.$$

Find two vectors in P and show that their sum is not in P.

The vectors $(1, 0, -2)$ and $(0, 1, -4)$ are both in P, but their sum, $(1, 1, -6)$, is not in P.

(a) Find a subset $W \subseteq \mathbb{R}^2$ where, for $v, w \in W$, $v+w \in W$, but cv is not necessarily in W.

We take as our subset all points in the nonnegative orthant, the set $\{(x, y) : x \geq 0, y \geq 0\}$. The sum of two points in W is again in W, but given a point in W, we cannot multiply it by a *negative* scalar to get another point in W.

(b) Find a subset $W \subseteq \mathbb{R}^2$ where, for $v, w \in W$, $cv \in W$, but $v + w$ is not necessarily in W.

We take the union of the two lines $\{(x, y) : x = y\}$ and $\{(x, y) : x = -y\}$. This set is closed under scalar multiplication — if we take a point on one of the lines and multiply it by a scalar, we get another point on the *same* line. However, if we add two points, each from one of the lines, say $(1, 1)$ and $(1, -1)$, we get $(1, 0)$, which is on neither line.

(8) **Let A and B be any $n \times n$ matrices. If $v \in N(B)$, show that $v \in N(A \cdot B)$. If A is invertible, show that if $v \in N(A \cdot B)$, then $v \in N(B)$.**

$$v \in N(B) \rightarrow Bv = 0 \rightarrow (A \cdot B)v = A(Bv) = 0 \rightarrow v \in N(A \cdot B)$$

If A is invertible, then A^{-1} exists, and we can write:

$$\begin{aligned} v \in N(A \cdot B) &\rightarrow (A \cdot B)v = 0 \\ &\rightarrow A^{-1}(A \cdot B)v = 0 \\ &\rightarrow (A^{-1}A)Bv = 0 \\ &\rightarrow IBv = 0 \\ &\rightarrow Bv = 0 \\ &\rightarrow v \in N(B) \end{aligned}$$