

ArsDigitaUniversity
Month2:DiscreteMathematics -ProfessorShaiSimonson

ProblemSet5 –CombinatoricsandCounting

1. Given n points in the plane with no three collinear,
 - a. How many different segments joining two points are there?
 - b. How many ways are there to choose a directed path of length two through three distinct points?
 - c. How many different triangles are there?
 - d. How many ways are there to choose 4 segments?
 - e. If you choose 4 segments at random, what is the chance that some three form a triangle?

2. Forty equally skilled teams play a tournament in which every team plays every other team exactly once, and there are no ties.
 - a. How many different games were played?
 - b. How many different possible outcomes for these games are there?
 - c. How many different ways are there for each team to win a different number of games?

3. Let $C(n, k)$ be the number of ways to choose k objects from a set of n . Prove by a combinatorial argument:
 - a. $C(n, 0) + C(n, 1) + \dots + C(n, n) = 2^n$.
 - b. $C(n, m)C(m, k) = C(n, k)C(n - k, m - k)$.
 - c. $C(n, n - k) = C(n, k)$.
 - d. $C^2(n, 0) + C^2(n, 1) + \dots + C^2(n, n) = C(2n, n)$. (Hint: Use c).

4. Prove the following combinatorial identities by formulae or mathematical induction
 - a. $C(n + 1, k + 1) = C(n, k + 1) + C(n, k)$.
 - b. $C(r, r) + C(r + 1, r) + C(r + 2, r) + \dots + C(n, r) = C(n + 1, r + 1)$.
 - c. Using the identity in 4b above, derive a formula for the sum $(1)(2)(3) + (2)(3)(4) + \dots + (n - 2)(n - 1)(n)$.

5. If you have $2n$ socks in a drawer, n white and n black, and you reach into to choose 2 socks at random,
 - a. How many ways are there to choose?
 - b. How many of these ways result in getting a pair of the same color?
 - c. Write a simple closed form formula in terms of n for the chance of choosing a matching pair of socks from a drawer with n white and n black socks.

6. A few short problems:
- How many ways are there to choose a president, vice president, secretary and treasurer from 9 people?
 - How many ways can 13 identical balls be distributed into 3 distinct boxes?
 - How many numbers greater than 3,000,000 can be formed from permutations of 1, 2, 2, 4, 6, 6, 6?
 - How many nine-digit numbers with twice as many odd digits as even digits? (leading zeros are allowed)
 - How many passwords can be created in the form $[A-Z][a-z]^9[0,1]^6$? (That is, a capital letter followed by 9 lowercase letters followed by 6 bits).
7. Poker:
- How many different 5-card Poker hands are there?
 - How many of these are 1 pair?
 - How many of these are a flush (all one suit)?
 - How many are a full house (3 of a kind and a pair)?
8. How many ways are there to distribute eight balls into six distinct boxes with at least one ball in each box if:
- The balls are identical?
 - The balls are distinct?
9. How many ways are there to distribute eight balls into six distinct boxes with at most four balls in the first two boxes if:
- The balls are identical?
 - The balls are distinct?
10. Fibonacci in Pascal's Triangle.
 Prove by induction that the n th Fibonacci number F_n equals $C(n,0) + C(n-1,1) + C(n-2,2) + \dots + C(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor)$. You should assume that $F_0 = F_1 = 1$.
11. Given a deck of 52 cards, how many ways are there to choose a set of four cards in order followed by one additional card? Three in order followed by two additional unordered cards? Two? One?
12. There's a news screen saver that displays a random rectangular piece of an n by n checkerboard.
- How many rectangles are there in a checkerboard of size 1? 2? 3? 4?
 - How many squares are there in a checkerboard of size 1? 2? 3? 4?
 - Guess a general formula for the number of squares and rectangles. Put each in closed form in terms of n .
 - Prove your formulas are true either by induction or using a combinatorial argument.
 - What's the chance that the rectangle displayed is a square? Give a simplified closed form in terms of n .
 - Although the number of squares and rectangles increase without bound as n increases, what happens to the ratio of squares to rectangles?